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ON ASYMPTOTIC PROPERTIES OF SYSTEMS WITH STRONG AND VERY STRONG HIGH-FREQUENCY EXCITATION

A. FIDLIN

LuK GS GmbH, 77815 *Buehl*, *Germany*. *E-mail*: *fidlin@luk.de*.

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The affect of high-frequency excitation on the low-frequency motions of dynamic systems is considered. It is suggested to differentiate between weak, strong and very strong high-frequency excitations. Several approaches and difficulties connected with the analysis of these systems are shown. Systems with strong excitation are examined in a general form. As an example, the responses of a one-degree-of-freedom system to strong and very strong, high-frequency external and parametric excitations are compared. It is indicated, how the results achieved could be generalized for mechanical systems with very strong excitation.

1. INTRODUCTION

Systems with high-frequency excitation have recently attracted great attention. The conventional technical approach to systems with high-frequency excitation, i.e., the excitation, where the frequency significantly exceeds the natural frequencies of the system, is that this excitation is almost unessential for the low-frequency motions of the system, because of its strong filtering properties. However, it is well known, that this point of view is not always correct. If high-frequency excitation is strong enough, it is able to significantly change the properties of the system with respect to slow motions. Numerous example of such systems can be found in the works of Bogoliubov and Mitropolskii, Nayfeh, Blekhman, Thomsen and others [1–8]. The main approach for the useful analysis of these systems is the separation of motions.

The separation of motions is one of the main ideas for asymptotic analysis of oscillating systems with small or large parameters. It is connected with the fact, that solutions of many types of dynamic systems can be represented as a superposition of fast oscillations and slow evolution. These slow motions are the main reason for interest in most applications. Powerful asymptotic methods, such as the averaging method of Bogoliubov and Mitropolskii [1], are in fact nothing but a practical realization of this idea. The same could be said about the direct separation of motions method, which originated in the works of Kapitsa [9] and was most generally formulated by Blekhman [3, 4]. The method of multiple scales, developed in the works of Naifeh [2, 10], is similar in substance to the averaging method.

In this work, three different levels of high-frequency excitation—weak, strong and very strong are analyzed. Several approaches and difficulties connected with the analysis of these systems are shown. Systems with strong excitation are examined in general form. As an example we compare the response of a one-degree-of-freedom system to strong and very strong high-frequency external and parametric excitation. Lastly, some ways are indicated,

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how the achieved results could be generalized for mechanical systems with very strong excitation.

2. CLASSIFICATION OF SYSTEMS WITH THE HIGH-FREQUENCY EXCITATION. SYSTEMS WITH WEAK EXCITATION

First of all, in this work we are going to consider mechanical systems, described by systems of second order differential equations in the following form:

$$\ddot{\mathbf{x}} = \omega^{\alpha}\mathbf{\Phi}(\mathbf{x}, \dot{\mathbf{x}}, t, \omega t), \tag{1}$$

where $\mathbf{x}$ is an n-dimensional vector of the generalized co-ordinates and $\dot{\mathbf{x}}$ is a vector of the generalized velocities, $\omega \gg 1$ is a big parameter. We take $\mathbf{\Phi}$ to be an n-dimensional vector of forces, which depends 2π-periodically on the fast time $\tau = \omega t$.

Depending on the magnitude of the integer parameter α we shall distinguish between systems with weak ($\alpha = 0$), strong ($\alpha = 1$) and very strong ($\alpha = 2$) excitation. Weak excitation is trivial, but it illustrates in the best way, how asymptotic methods are used for such problems. Let us use, for example, the averaging technique and rewrite our system as a system of $2n$ first order equations:

$$\dot{\mathbf{x}} = \mathbf{y}, \quad \dot{\mathbf{y}} = \mathbf{\Phi}(\mathbf{x}, \mathbf{y}, t, \tau). \tag{2}$$

Converting to the fast time τ as an independent variable, we get

$$\mathbf{x}' = \varepsilon\mathbf{y}, \quad \mathbf{y}' = \varepsilon\mathbf{\Phi}(\mathbf{x}, \mathbf{y}, t, \tau), \quad \varepsilon = \frac{1}{\omega} \ll 1, \quad t' = \varepsilon. \tag{3}$$

If our function $\mathbf{\Phi}$ is smooth enough, we get the standard form system of Bogoliubov and Mitropolskii and can directly apply the averaging procedure. The equation of the first approximation takes the form

$$\ddot{\boldsymbol{\xi}} = \langle \mathbf{\Phi}(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}, t, \tau) \rangle. \tag{4}$$

Here $\langle \mathbf{\Phi} \rangle$ is the average of $\mathbf{\Phi}$ with respect to the fast time τ, and $(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$ are asymptotically close to the solution of the original system $(\mathbf{x}, \dot{\mathbf{x}})$ for the time interval

$$\tau = O(\omega) \text{ or } t = O(1).$$

The obtained solution is a superposition of small high-frequency oscillations and slow evolution of the system for both the generalized co-ordinates and the generalized velocities. This case is well known, so more complicated case of the systems with strong excitation should be considered.

3. SYSTEM WITH STRONG EXCITATION

Particular systems with strong excitation are usual in various applications. A lot of them are studied in detail in the works of Blekhman [3, 4], who considered these systems in the following form:

$$\ddot{\mathbf{x}} = F(\mathbf{x}, \dot{\mathbf{x}}, t, \tau) + \omega\mathbf{\Phi}(\mathbf{x}, t, \tau) \tag{5}$$

for which some very efficient asymptotic methods are established. However, the absence of $\dot{\mathbf{x}}$ in the strong part of the excitation $\mathbf{\Phi}$ is significant for these special methods.

Usually, these systems are used if we are interested in analyzing the motion of a machine supposing, that the inertia of its housing is significantly larger then the inertia of its moving parts. If the mass or inertia of the mechanism's moving parts is not small (for example, the type of modern mechanisms often found in crank gears or vane pumps), equations of motion containing quick oscillating inertia coefficients will be obtained. These equations contain large, fast oscillating terms. These depend not only on the generalized co-ordinates but also on the system's generalized velocities.

Another example of systems with strong excitation depending on the first derivative of the unknown function, appears if we investigate vibrations or wave propagation in inhomogenious media. For example, the longitudinal waves in a rod with a periodic or quasi-periodic structure. In this case the typical equations with strong excitation of general form appear naturally with respect to the spatial co-ordinates. Equations with the slow modulated, high-frequency excitation, which we are going to analyze in sections 4 and 5 of this paper, are first of all typical for this group for applications. Firstly, a general mathematical approach to the systems with strong excitation is given, described by a system of differential equations as follows:

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t, \tau) + \omega\mathbf{\Phi}(\mathbf{x}, \dot{\mathbf{x}}, t, \tau), \tag{6}$$

where here $\mathbf{F}$ is a weak part and $\mathbf{\Phi}$ is a strong part of the high-frequency excitation. In order to analyze this system, we are going to use the multiple-scale technique. This is because it is easier from the technical point of view. The averaging method could also be used, which leads to identical results and gives some advantages in validation (see reference [11]).

In order to apply the multiple-scales technique, we have to convert from a system of ordinary differential equations (6) to a system with partial derivatives and two independent variables t and τ:

$$\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2} + 2\omega\frac{\partial^2\boldsymbol{\varphi}}{\partial t\partial\tau} + \omega^2\frac{\partial^2\boldsymbol{\varphi}}{\partial\tau^2} = \mathbf{F}\left(\boldsymbol{\varphi}, \frac{\partial\boldsymbol{\varphi}}{\partial t} + \omega\frac{\partial\boldsymbol{\varphi}}{\partial\tau}, t, \tau\right) + \omega\mathbf{\Phi}\left(\boldsymbol{\varphi}, \frac{\partial\boldsymbol{\varphi}}{\partial t} + \omega\frac{\partial\boldsymbol{\varphi}}{\partial\tau}, t, \tau\right). \tag{7}$$

The relationship between equations (6) and (7) is given by a condition, that if $\varphi(t, \tau)$ is a solution of equation (7), then this solution taken along the straight line $\tau = \omega t$, i.e., $\mathbf{x} = \boldsymbol{\varphi}(t, \omega t)$ is a solution of equation (6). In other words, system (7) is more general than equation (6), so we are free in choice of boundary conditions for this system. The only restriction is that the straight line $\tau = \omega t$ should be in the inner part of the considered area. We require $\boldsymbol{\varphi}(t, \tau)$ to be a 2π-periodic function of τ and try to find $\boldsymbol{\varphi}(t, \tau)$ as a formal asymptotic expansion in terms of the small parameter ε:

$$\boldsymbol{\varphi}(t, \tau) = \boldsymbol{\psi}_0(t, \tau) + \varepsilon\boldsymbol{\psi}_1(t, \tau) + \varepsilon^2\boldsymbol{\psi}_2(t, \tau) + \cdots . \tag{8}$$

Substituting this expansion into equation (7) and balancing the terms with equal orders of ε we obtain

$$\varepsilon^{-2}: \quad \frac{\partial^2\boldsymbol{\psi}_0}{\partial\tau^2} = 0, \tag{9}$$

$$\varepsilon^{-1}: \quad \frac{\partial^2\boldsymbol{\psi}_1}{\partial\tau^2} + 2\frac{\partial^2\boldsymbol{\psi}_0}{\partial t\partial\tau} = \mathbf{\Phi}\left(\boldsymbol{\psi}_0, \frac{\partial\boldsymbol{\psi}_1}{\partial\tau} + \frac{\partial\boldsymbol{\psi}_0}{\partial t} + \omega\frac{\partial\boldsymbol{\psi}_0}{\partial\tau}, t, \tau\right), \tag{10}$$

$$\varepsilon^0: \quad \frac{\partial^2 \boldsymbol{\psi}_2}{\partial \tau^2} + 2\frac{\partial^2 \boldsymbol{\psi}_1}{\partial t \partial \tau} + \frac{\partial^2 \boldsymbol{\psi}_0}{\partial t^2} = \mathbf{F} + \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}} \boldsymbol{\psi}_1 + \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \left(\frac{\partial \boldsymbol{\psi}_1}{\partial t} + \frac{\partial \boldsymbol{\psi}_2}{\partial \tau} \right). \tag{11}$$

The last step must be justified, because the second argument of all the functions on the right-hand sides of the equations in given as

$$\frac{\partial \boldsymbol{\psi}_1}{\partial \tau} + \frac{\partial \boldsymbol{\psi}_0}{\partial t} + \omega \frac{\partial \boldsymbol{\psi}_0}{\partial \tau}. \tag{12}$$

This expression can take values of the magnitude order of our big parameter ω. So it can create in our equation terms of any order, depending on the kind of dependence of $\boldsymbol{\Phi}$ from $\dot{\mathbf{x}}$. However, if we require $\boldsymbol{\Phi}$, as usual, to be a bounded function in the vicinity of the solution of the averaged system, we reduce the problem to the *a posteriori* check of our assumptions about the magnitude order of $\dot{\mathbf{x}}$ in the vicinity of the found solution.

However, in our case the problem is insignificant. The general solution of equation (9) has the following form:

$$\boldsymbol{\psi}_0(t, \tau) = \mathbf{X}(t) + \mathbf{A}(t)\tau, \tag{13}$$

According to the periodicity of $\boldsymbol{\psi}_0$, we get

$$\mathbf{A}(t) = 0.$$

Hence, $\boldsymbol{\psi}_0 = \mathbf{X}(t)$. This depends only on the slow time t and the large terms in equations (11) and (12) disappear automatically. However, we shall return to this analysis later, considering systems with very strong excitation. For these systems it will be of paramount importance.

The objective of the following analysis is to discover differential equations for the still unknown function $\mathbf{X}(t)$, which do not contain the fast time τ.

Equation (10) after substituting in it the solution of equation (9) takes the form

$$\frac{\partial^2 \boldsymbol{\psi}_1}{\partial \tau^2} = \boldsymbol{\Phi}\left(\mathbf{X}, \dot{\mathbf{X}} + \frac{\partial \boldsymbol{\psi}_1}{\partial \tau}, t, \tau \right). \tag{14}$$

It is natural to call this equation "The Equation of Fast Motions". It is a differential equation with only partial derivatives in respect τ. That is why we can take $\mathbf{X}$, $\dot{\mathbf{X}}$ and t to be constant parameters during solving equation (14).

The main assumption of this analysis is that we know the general 2π-periodic with respect to τ solution of the system of n first order differential equations

$$\frac{\partial u}{\partial t} = \boldsymbol{\Phi}(\mathbf{X}, \mathbf{u}, t, \tau) \tag{15}$$

in which X and t are taken to be constant:

$$\mathbf{u} = \mathbf{U}(\mathbf{X}, t, \tau, \mathbf{C}).$$

Here C is a vector of arbitrary constants, which can be found due to the condition of the periodicity of $\boldsymbol{\psi}_1$, i.e., we have to annihilate the average of $\partial \boldsymbol{\psi}_1 / \partial \tau$:

$$\langle \mathbf{U}(\mathbf{X}, t, \tau, \mathbf{C}) \rangle = \dot{\mathbf{X}}. \tag{16}$$

If we succeed in solving this equation in respect to **C**, we can rewrite the solution of equation (14) as follows:

$$\mathbf{u} = \mathbf{U}(\mathbf{X}, \dot{\mathbf{X}}, t, \tau), \quad \boldsymbol{\psi}_1 = (t, \tau) + \mathbf{X}_1(t), \tag{17}$$

$$\boldsymbol{\Psi}(t, \tau) = \int_0^{\tau} (\mathbf{U} - \dot{\mathbf{X}})\, d\tau. \tag{18}$$

The new unknown function $\mathbf{X}_1(t)$ is a small slow correction to the main slow part of the solution $\mathbf{X}(t)$. System (15) is significantly simpler than the original system (6), because its order is twice lower and we can take all the functions of slow time t to be constant. In other words here, we are not interested in the slow evolution of the system, but only in its high-frequency oscillations.

Let us move on now to the equation of second approximation (11) as follows:

$$\frac{\partial^2 \boldsymbol{\psi}_2}{\partial \tau^2} = \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \frac{\partial \boldsymbol{\psi}_2}{\partial \tau} + \mathbf{F} + \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}} \boldsymbol{\psi}_1 + \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \frac{\partial \boldsymbol{\psi}_1}{\partial t} - 2 \frac{\partial^2 \boldsymbol{\psi}_1}{\partial t \partial \tau} - \ddot{\mathbf{X}}. \tag{19}$$

This is a system of n first order linear inhomogeneous equations with periodic coefficients. The unknown function is $\partial \boldsymbol{\psi}_2 / \partial \tau$. The necessary condition for the existence of its periodic solutions is well known:

$$\left\langle \mathbf{W}_*^{\mathrm{T}} \left\{ F + \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}} \boldsymbol{\psi}_1 + \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \frac{\partial \boldsymbol{\psi}_1}{\partial t} - 2 \frac{\partial^2 \boldsymbol{\psi}_1}{\partial t \partial \tau} - \ddot{\mathbf{X}} \right\} \right\rangle = 0. \tag{20}$$

Here $\mathbf{W}_*$ is the fundamental matrix of solutions for a system conjugated to the homogeneous part of equation (19):

$$\frac{\partial \mathbf{W}_*}{\partial \tau} = -\left(\frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \right)^{\mathrm{T}} \mathbf{W}_* \tag{21}$$

In order to get the final form of equation (20), let us notice several identities:

$$\left\langle \mathbf{W}_*^{\mathrm{T}} \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \right\rangle = 0, \quad \left\langle \mathbf{W}_*^{\mathrm{T}} \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}} \right\rangle = 0, \quad \left\langle \mathbf{W}_*^{\mathrm{T}} \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\mathbf{x}}} \frac{\partial \boldsymbol{\Psi}}{\partial t} \right\rangle = \left\langle \mathbf{W}_*^{\mathrm{T}} \frac{\partial^2 \boldsymbol{\Psi}}{\partial t \partial \tau} \right\rangle. \tag{22}$$

Due to the identities, one can show, that equation (20) does not contains $X_1(t)$, and reduce this equation as follows:

$$\langle \mathbf{W}_*^{\mathrm{T}} \rangle \ddot{\mathbf{X}} = \left\langle \mathbf{W}_*^{\mathrm{T}} \left\{ \mathbf{F} + \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}} \boldsymbol{\Psi} - \frac{\partial^2 \boldsymbol{\Psi}}{\partial t \partial \tau} \right\} \right\rangle. \tag{23}$$

Lastly, the function $\boldsymbol{\Psi}$ depends on t both direct and indirect through functions $\mathbf{X}(t)$ and $\dot{\mathbf{X}}(t)$. Under $\partial/\partial t$ we understand here the full partial derivative with respect to t. Taking this into consideration and using the partial derivatives we will obtain the final explicit form of equation (23):

$$\mathbf{M}(\mathbf{X}, \dot{\mathbf{X}}, t) \ddot{\mathbf{X}} = \mathbf{V}(\mathbf{X}, \dot{\mathbf{X}}, t). \tag{24}$$

Here,

$$\mathbf{M}(\mathbf{X},\dot{\mathbf{X}},t)=\left\langle \mathbf{W}_*^{\mathrm{T}}\frac{\partial\mathbf{U}}{\partial\dot{\mathbf{X}}}\right\rangle,\quad \mathbf{V}(\mathbf{X},\dot{\mathbf{X}},t)=\left\langle \mathbf{W}_*^{\mathrm{T}}\left\{\frac{\partial\mathbf{\Phi}}{\partial\mathbf{x}}\mathbf{\Psi}-\frac{\partial\mathbf{U}}{\partial\mathbf{X}}\dot{\mathbf{X}}-\frac{\partial\mathbf{U}}{\partial t}\right\}\right\rangle$$

$$+\langle\mathbf{W}_*^{\mathrm{T}}\mathbf{F}(\mathbf{X},\mathbf{U},t,\tau)\rangle. \tag{25}$$

Equations (24) do not contain fast time and determine slow evolution of the solutions of the original system (6). That is why they could be called “Equations of Slow Motions”. Function $\mathbf{V}(\mathbf{X},\dot{\mathbf{X}},t)$ is natural to call “vibration force” and Matrix $\mathbf{M}(\mathbf{X},\dot{\mathbf{X}},t)$ can be interpreted as a matrix of the averaged system’s efficient mass with respect to slow motions. This matrix depends on the solution of the equations of fast motions, i.e., on the amplitude of fast excitation.

If we have initial conditions for the original system

$$x|_{t=0}=x_0,\quad \dot{x}|_{t=0}=v_0, \tag{26}$$

then the initial conditions for the averaged system can be calcualted as follows:

$$\mathbf{X}|_{t=0}=\mathbf{x}_0,\quad \dot{\mathbf{X}}|_{t=0}=\mathbf{v}_0-\left.\frac{\partial\mathbf{\Psi}}{\partial\tau}\right|_{t=0,\tau=0}. \tag{27}$$

If function $\mathbf{\Phi}$ does not depend on $\dot{\mathbf{x}}$, system (24) goes over into equations, well known, for example, from the works of Blekhman [4]:

$$\ddot{\mathbf{X}}=\left\langle \mathbf{F}+\frac{\partial\mathbf{\Phi}}{\partial x}\mathbf{\Psi}\right\rangle. \tag{28}$$

4. AN EXAMPLE OF SYSTEMS WITH STRONG EXCITATION

Let us give a simple example, illustrating several properties of systems with strong excitation.

An example, showing the main property of systems to be considered: the effective mass of the averaged system can differ from its real mass and depends on the amplitude of the excitation can be found in reference [11]. We are going now to consider another example, which has simple mechanical origins.

The equations with strong excitation appear naturally, as it was mentioned above, in systems with oscillating inertia coefficients and in continuous systems with periodic structure. Let us take the simplest example of such an equation:

$$\left(\frac{\dot{\mathbf{x}}}{1+c(t)\cos\omega t}\right)^{\bullet}+\mathbf{x}+\alpha\mathbf{x}^3=\omega f(t)\sin(\omega t+\theta). \tag{29}$$

We can rewrite this equation as follows:

$$\ddot{x}=-(\mathbf{x}+\alpha\mathbf{x}^3)(1+c\cos\tau)+\frac{\dot{\mathbf{x}}\dot{c}\cos\tau}{(1+c\cos\tau)}+\omega f\sin(\tau+\theta)(1+c\cos\tau)-\omega\frac{\dot{\mathbf{x}}c\sin\tau}{(1+c\cos\tau)}. \tag{30}$$

This is a system with strong excitation. Referring back to the general form (6) we can sign

$$\mathbf{F} = -(\mathbf{x} + \alpha\mathbf{x}^3)(1 + c\cos\tau) + \frac{\dot{\mathbf{x}}\dot{c}\cos\tau}{(1 + c\cos\tau)},$$
$$\mathbf{\Phi} = f\sin(\tau + \theta)(1 + c\cos\tau) - \frac{\dot{\mathbf{x}}c\sin\tau}{(1 + c\cos\tau)}. \tag{31}$$

The corresponding equation of fast motions has the form

$$\frac{\partial\mathbf{u}}{\partial\tau} = -\frac{\mathbf{u}c\sin\tau}{(1 + c\cos\tau)} + f\sin(\tau + \theta)(1 + c\cos\tau). \tag{32}$$

Its solution, which fulfils the condition $\langle\mathbf{u}\rangle = \dot{\mathbf{X}}$ is

$$\mathbf{u} = (\dot{\mathbf{X}} + \tfrac{1}{2}fc\cos\theta - f\cos(\tau + \theta))(1 + c\cos\tau). \tag{33}$$

Solution of the system conjugated to the homogeneous part of the equation of second approximation

$$\frac{\partial\mathbf{W}_*}{\partial\tau} = \frac{c\mathbf{W}_*\sin\tau}{(1 + c\cos\tau)} \tag{34}$$

is also simple to find

$$\mathbf{W}_* = \frac{1}{(1 + \cos\tau)}. \tag{35}$$

Averaging the corresponding terms we shall obtain

$$\mathbf{M} = \left\langle \mathbf{W}_* \frac{\partial\mathbf{u}}{\partial\dot{\mathbf{x}}} \right\rangle = 1,$$
$$\mathbf{V} = \left\langle \mathbf{W}_* \left\{ \mathbf{F}(\mathbf{X}, \mathbf{U}, t, \tau - \frac{\partial\mathbf{u}}{\partial t} \right\} \right\rangle = -\mathbf{X} - \alpha\mathbf{X}^3 - \tfrac{1}{2}(fc)^{\bullet}\cos\theta.$$

Finally, we obtain the equation of slow motions

$$\ddot{\mathbf{X}} + \mathbf{X} + \alpha\mathbf{X}^3 = -\tfrac{1}{2}(fc)^{\bullet}\cos\theta. \tag{36}$$

This equation has several interesting peculiarities. Firstly, in this case, there is no transformation of the effective mass of this system or its natural frequency. Instead of it we have another interesting phenomenon—transformation of the slow excitation's character. There are both external and parametric high-frequency excitations in the original system. If the high-frequency excitations are slowly modulated through the functions $f(t)$ and $c(t)$, it means, that we do not only deal with the high-frequency but also with the low-frequency parametric and external excitations of the original system. In the equation of slow motions the parametric excitation disappeared. The external excitation is transformed in an

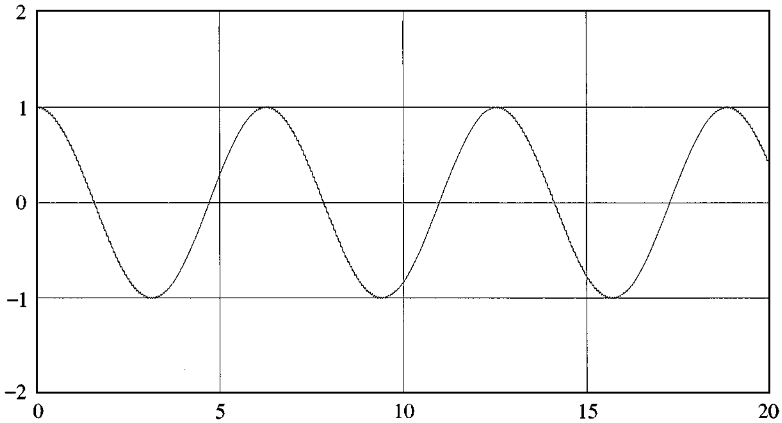

Figure 1. System with strong excitation: (—) travel.

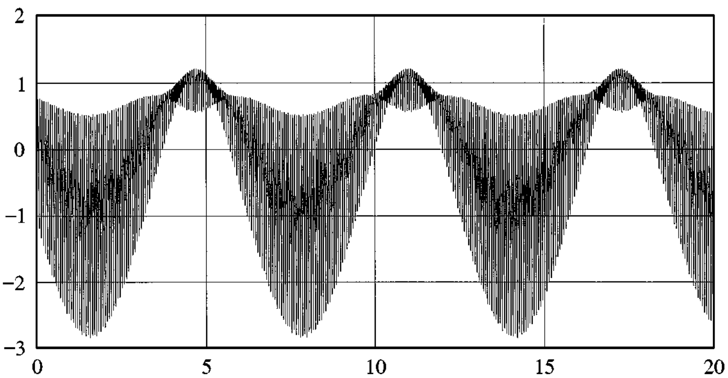

Figure 2. System with strong excitation: (—) velocity.

unexpected way. It got to be proportional to the first derivative of the product of the slow variable amplitudes of both external and parametric high-frequency excitations.

The properties of the averaged system are illustrated through the following figures, obtained through the numeric simulation of the full equation (29). Figures 1 and 2 show the solution for the following values of parameters:

$$\alpha = 0, \quad \omega = 100, \quad \theta = 0, \quad f = 1 + \tfrac{1}{2}\sin t;, \quad c = \frac{1}{3(1 + \frac{1}{2}\sin t)}.$$

This figure illustrates the typical character of the solutions of systems with strong excitation, which is a superposition of slow motion and fast oscillations. Important here is, that the amplitude of the fast oscillations is small in relationship to the amplitude of the slow changes of the generalized co-ordinates. The amplitude of the fast velocity oscillations is comparable with the amplitude of its slow evolution.

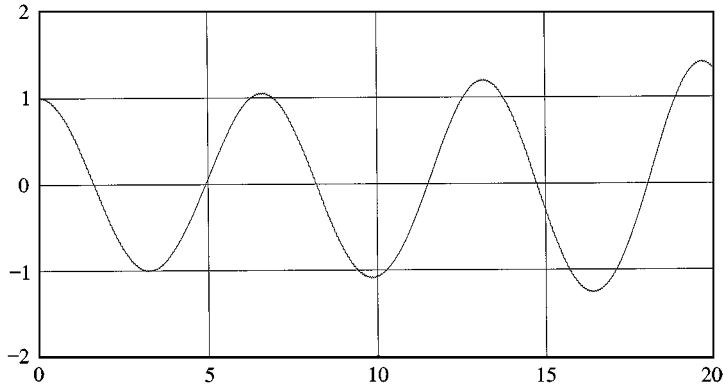

Figure 3. Slow resonance in the system with strong excitation: (—) travel.

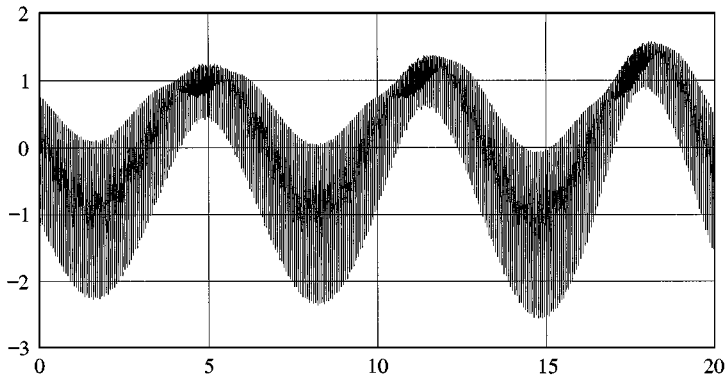

Figure 4. Slow resonance in the system with strong excitation: (—) velocity.

In this case, we have $(fc)^{\bullet} = 0$. As it can be seen, in this case we have only free oscillations of the averaged system. Figures 3 and 4 show the results of the simulation for the case $(fc)^{\bullet} \neq 0$:

$$f = 1, \quad c = \frac{1}{3(1 + \frac{1}{2}\sin t)}.$$

In this case, according to the prediction, we have a typical picture of the external non-parametric resonance with the linear increasing amplitude of slow oscillations. In these cases there is no visible difference between the two predictions. Figure 5 shows the comparison of analytic and numeric solutions for $\omega = 5$. We can see, that, although in this case the small parameter is not small enough, the asymptotic solution still gives the qualitative character of the system's movement. However, the quantitative differences are significant.

5. SYSTEMS WITH VERY STRONG EXCITATION

We should now consider the significantly more difficult case of systems with very strong excitation. Initially, the same system with one degree of freedom should be considered when

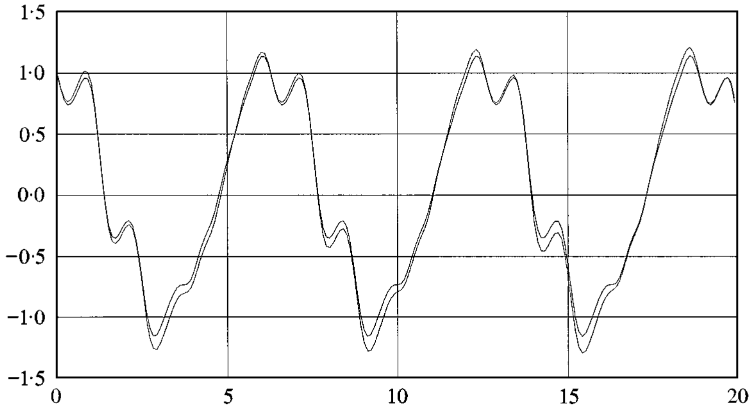

Figure 5. Comparison of asymptotic and numeric solution. Travels (—) asymptotic solution; (—) numeric experiment.

the external excitation is very strong:

$$\left(\frac{\dot{\mathbf{x}}}{1 + c(t)\cos\tau}\right)^{\boldsymbol{\cdot}} + x\alpha x^3 = \omega^2 f(t)\sin(\tau + \theta). \tag{37}$$

This equation can be rewritten as follows:

$$\ddot{x} = \omega^2 f \sin(\tau + \theta)(1 + c\cos\tau) - \omega\frac{\dot{x}c\sin\tau}{(1 + c\cos\tau)} + \frac{\dot{x}\dot{c}\cos\tau}{(1 + c\cos\tau)} - (x + \alpha x^3)(1 + c\cos\tau). \tag{38}$$

We are going to apply the multiple-scales technique and try to find a solution of equation (38) in the form of equation (8). In this case, however, we shall get an important equation already balancing the terms to the magnitude order of ω^2:

$$\frac{\partial^2\psi_0}{\partial\tau^2} = f\sin(\tau + c\cos\tau)(1 + c\cos\tau) - \frac{c\sin\tau}{(1 + c\cos\tau)}\frac{\partial\psi_0}{\partial t}. \tag{39}$$

It can be noticed, that, denoting $\partial\psi_0/\partial\tau$ through u, the previously analyzed equation of fast motions of a system with strong excitation is obtained. Its solution is already known. The only difference is that now its average has to vanish, i.e.,

$$\langle u\rangle = 0.$$

The corresponding solution is

$$\begin{aligned} u &= (\tfrac{1}{2}fc\cos\theta - f\cos(\tau + \theta))(1 + c\cos\tau), \\ \psi_0 &= X_0(t) - f\sin(\tau + \theta) - \tfrac{1}{4}fc\sin(2\tau + \theta) + \tfrac{1}{2}fc^2\sin\tau\cos\theta. \end{aligned} \tag{40}$$

Our objective is to find an equation determining $X_0(t)$, which does not contain the fast time τ.

The equation of the first approximation should now be considered:

$$\frac{\partial^2\psi_1}{\partial\tau^2} + 2\frac{\partial^2\psi_0}{\partial t\partial\tau} = -\frac{c\sin\tau}{(1 + c\cos\tau)}\left(\frac{\partial\psi_0}{\partial t} + \frac{\partial\psi_1}{\partial\tau}\right) + \frac{\dot{c}\cos\tau}{1 + c\cos\tau}\frac{\partial\psi_0}{\partial\tau}. \tag{41}$$

We denote

$$\frac{\partial\psi_0}{\partial t} + \frac{\partial\psi_1}{\partial\tau} = v.$$

For the new variable v, a linear equation with periodic coefficients will be obtained:

$$\frac{\partial v}{\partial\tau} = -\frac{c\sin\tau}{(1 + c\cos\tau)}v - (1 + c\cos\tau)\frac{\partial}{\partial t}\left(\frac{u}{1 + c\cos\tau}\right). \tag{42}$$

This equation does not contain X_0. Meaning, this equation cannot be used to determine the slow part of the solution. However, it gives us a restriction, which is able to be fulfilled by functions $f(t)$ and $c(t)$. This restriction is necessary for the solutions of this type to exit. This condition is not difficult to find if we, as usual, require the periodicity of the solution of equation (42). This condition has the following form:

$$\left\langle\frac{\partial}{\partial t}\left(\frac{u}{1 + c\cos\tau}\right)\right\rangle = \langle\tfrac{1}{2}(fc)^{\bullet}\cos\theta - f\cos(\tau + \theta)\rangle = \tfrac{1}{2}(fc)^{\bullet}\cos\theta = 0.$$

So the functions $f(t)$ and $c(t)$ have to fulfil the equation

$$(fc)^{\bullet}\cos = 0. \tag{43}$$

Assuming, that this condition is fulfilled, we can then find the function v if we rewrite equations (42) in a simple form

$$\frac{\partial}{\partial\tau}\left(\frac{v}{1 + c\cos\tau}\right) = -\frac{\partial}{\partial t}\left(\frac{u}{1 + c\cos\tau}\right) = f\cos(\tau + \theta).$$

Its solution, which fulfils the periodicity condition for ψ_1,

$$\langle v\rangle = \dot{X}_0, \tag{44}$$

is

$$v = (\dot{X}_0 - \tfrac{1}{2}\dot{f}c\sin\theta + \dot{f}\sin(\tau + \theta))(1 + x\cos\tau),$$

$$\psi_1 = -\left(2 - \frac{c^2}{2}\right)\dot{f}\cos(\tau + \theta) + \frac{1}{4}\dot{f}c\sin(2\tau + \theta) + \dot{X}_0 c\sin\tau \tag{45}$$
$$+ fc\dot{c}\cos\theta\cos\tau + X_1(t).$$

Finally, the equation of the second approximation must be considered:

$$\frac{\partial^2\psi_2}{\partial\tau^2} + 2\frac{\partial^2\psi_1}{\partial t\partial\tau} + \frac{\partial^2\psi_0}{\partial t^2} = -\frac{c\sin\tau}{(1 + c\cos\tau)}\left(\frac{\partial\psi_1}{\partial t} + \frac{\partial\psi_2}{\partial\tau}\right) + \frac{\dot{c}\cos\tau}{(1 + c\cos\tau)}\left(\frac{\partial\psi_0}{\partial t} + \frac{\partial\psi_1}{\partial\tau}\right)$$
$$- (\psi_0 + \alpha\psi_0^3)(1 + c\cos\tau). \tag{46}$$

A new variable will now be introduced:

$$q = \frac{\partial \psi_2}{\partial \tau} + \frac{\partial \psi_1}{\partial t}.$$

q will be defined by the following equation:

$$\frac{\partial q}{\partial \tau} = -\frac{c \sin \tau}{(1 + c \cos \tau)} q - (1 + c \cos \tau)\left\{\psi_0 + \alpha \psi_0^3 - \frac{\partial}{\partial t}\left(\frac{v}{1 + c \cos \tau}\right)\right\}. \tag{47}$$

Requiring, this equation to have a periodic solution, we obtain the equation of slow motions,

$$\langle -\ddot{X}_0 + \tfrac{1}{2}(\dot{f}c)^{\bullet} - \ddot{f} \sin(\tau + \theta) + \psi_0 + \alpha \psi_0^3 \rangle = 0.$$

After calculating the average, we find the final form of this equation,

$$\begin{aligned} \ddot{X}_0 + \left\{1 + \frac{3}{2}\alpha f^2 \left(1 + \frac{1}{16}c^2 + \frac{1}{4}c^4 \cos^2\theta - c^2 \cos^2\theta\right)\right\} X_0 + \alpha X_0^3 \\ = \frac{1}{2}(\dot{f}c)^{\bullet} \sin\theta + \frac{3}{16}\alpha f^3 c \sin\theta\left(1 - \frac{1}{4}c^4 \cos^2\theta\right). \end{aligned} \tag{48}$$

The main properties of this system are very similar to those of the system with strong excitation. The frequency of the free oscillations of the averaged system depends on the amplitude of the high-frequency excitation. If both external and parametric excitations are slowly modulated, we can find both the parametric and external slow excitations of the averaged system. However, they are significantly changed. For the slow parametric excitation to exist, the system has to be non-linear ($\alpha \neq 0$) and it is therefore necessary to have a modulated external excitation ($f \neq 0$).

If we take the simplest linear situation, we obtain an equation, which is very similar to the situation of strong excitation:

$$\ddot{X}_0 + X_0 = \tfrac{1}{2}(\dot{f}c)^{\bullet} \sin\theta. \tag{49}$$

The only difference is that the external slow excitation depends on the second derivative of the slow modulation. Another point is, that in this case the excitation is proportional to the sines of the phase difference between the external and the parametric high-frequency excitations—not to the cosines as in the previous case.

It should be noted that a particular example of this system, without parametric excitation, was considered by Nayfeh and Nayfeh [12]. Equation (48) conforms completely to their results.

Figure 6 represents the direct numeric simulation results with regard to the full equations (37). The calculation was carried out using the following parameters:

$$\alpha = 0, \quad \theta = 0, \quad \omega = 100, \quad f = 1 + \tfrac{1}{2}\sin t, \quad c = \frac{1}{3(1 + \frac{1}{2}\sin t)}.$$

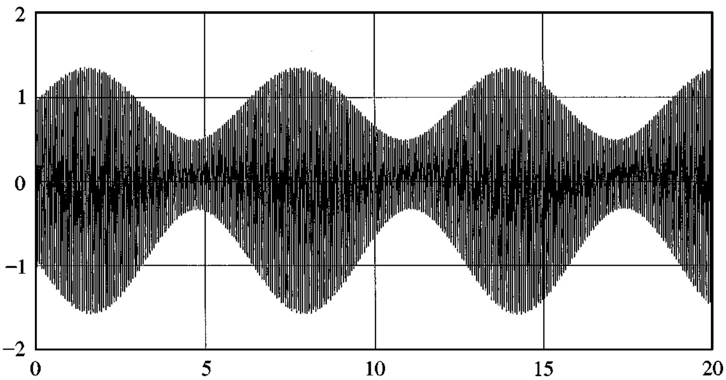

Figure 6. System with very strong excitation. Weak response: (—) travel.

It can be noted that the solution now is a superposition of fast oscillations and comparable large slow motions—even at the level of the generalized co-ordinates.

At the beginning of this analysis it was assumed that the necessary condition for the existence of such solutions was represented as follows:

$$(fc)^{\bullet} \cos\theta \neq 0.$$

However, the equation must be analyzed in the event of this condition being unfulfilled.

No solution with an amplitude of order 1 exists in this case. However, there are solutions with bigger amplitudes. In order to explain this, the simplest linear situation will be taken and the scale of our variable x changed:

$$\mathbf{x} = \omega \mathbf{z}. \tag{50}$$

For the new variable z an equation will be obtained as follows:

$$\ddot{z} = -z(1 + c\cos\tau) + \frac{\dot{z}\dot{c}\cos\tau}{(1 + c\cos\tau)} + \omega f \sin(\tau + \theta)(1 + c\cos\tau) - \omega\frac{\dot{z}c\sin\tau}{(1 + c\cos\tau)}. \tag{51}$$

This is only the linear variant of the previously analyzed equation with strong excitation (30). Its solutions are known.

Hence, in systems with very strong excitation we should distinguish between two types of solutions. These will be referred to as "weak response" and "strong response". The strong response with a slow amplitude, significantly greater than 1, can exist under more general conditions than the weak response. If some additional restrictions are placed on the character of the excitation's modulation, the weak response appears side by side with the strong response. Its amplitude has the magnitude order of 1. Figure 7 represents the direct numeric simulation results with regard to the full equations (45). The calculation was carried out using the following parameters:

$$\alpha = 0, \quad \theta = 0, \quad \omega = 100, \quad f = 1, \quad c = \frac{1}{3(1 + \frac{1}{2}\sin t)}.$$

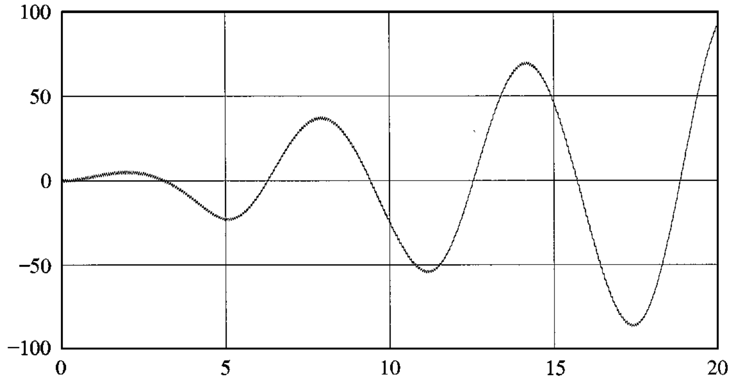

Figure 7. System with very strong excitation. Strong response: (—) travel.

In this situation, the weak response does not exist. The reaction of the system is typical for the external resonance, as predicted by the equation of slow motions (36). The amplitude of the oscillations has from the very beginning the magnitude order ω. The described properties of this particular system with very strong excitation is not a coincidence. Unfortunately, there are significant difficulties in the asymptotic analysis of general systems with very strong excitation. These difficulties are connected with the problem, that instead of terms, containing the generalized velocities $\dot{\mathbf{x}}$, terms such as those in equation (12) with $\partial\psi_0/\partial\tau \neq 0$ will be obtained. If the right-hand sides of the equations depend on $\dot{\mathbf{x}}$ arbitrary, these terms give rise to terms of the arbitrary asymptotic order. This situation forces a special form of system to be chosen. It is important to look at how the right-hand sides of the equations depend on the generalized velocities. For example, the previous analysis can be generalized for mechanical systems:

$$\ddot{\mathbf{x}}_k = \omega^2 \Phi_2^k(t, \tau) + \omega \sum_{i=1}^{n} \Phi_{1i}^k(t, \tau)\dot{x}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \Phi_{0ij}^k(t, \tau)\dot{x}_i\dot{x}_j$$

$$+ \sum_{i=1}^{n} F_{1i}^k(t, \tau)\dot{x}_i + F^k(x, t, \tau), \tag{52}$$

$$k = 1, \ldots, n.$$

The detailed analysis of such systems is beyond the limits of this paper. However, it must be noted, that in this situation there are also two types of solutions, which correspond to whether strong or weak response of the system to very strong excitation.

6. CONCLUSIONS

In this paper, the dynamics of systems are considered. These are described through systems of second order differential equations. These systems are subjected to a strong (magnitude order of big parameter ω) or very strong (magnitude order of ω^2) high frequency excitation. Equations of this type occur naturally if we analyze, for example, dynamics of a machine housing, where inertia depends significantly on the position of the moving parts of the internal mechanism. Typical applications are a crank mechanism or a vane pump. Another, perhaps even more important example of systems, with strong excitation depending on the first derivative of the unknown function, appears if we investigate vibrations or wave propagation in the inhomogenious media. For example, the longitudinal waves in a rod with a periodic or quasi-periodic structure. For systems with strong excitation, an averaging procedure is shown, which allows a general study of such systems. An example of a mechanical system with both parametric excitation and external (non-parametric) excitation shows one of the typical properties of such system—that modulated, high-frequency, strong parametric excitation in the presence of non-modulated, external excitation can turn into slow external excitation. It can lead to resonance in the system.

The same example is considered with very strong excitation. This illustrates the main pecularities of the systems:

- there are two types of solutions, which can be called "weak response" and "strong response";
- the first type of solution is more general; some additional conditions in the modulation of excitation must be fulfilled for the of second type solutions to exist;
- other properties are similar to those systems, which have strong excitation, but both external and parametric low-frequency excitations are possible in the averaged systems.

These results can be generalized for mechanical systems, where kinetic energy depends on quadratic and linear forms of the generalized velocities.

Comparison of the asymptotic analytic solutions with numeric simulations of the full differential equations (not averaged), illustrates the described phenomena.

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